The state variables of the system are the numerical quantities memorized by the system that comprise the state. In Fig. 2, $v_1(n),...,v_M(n)$ are the internal variables which comprise the state variables for this system [5].

We have,

$$v_i(n+1)=v_{i+1}(n) \quad (2)$$

$$v_M(n+1)=v_{M+1}(n)=x(n)+w(n)+a(1)v_M(n)+ a(2)v_{M-1}(n)+....+a(M)v_1(n)=x(n)+w(n)+\sum_{i=1}^{M}a(i)v_{M-i+1}(n) \quad (3)$$

Eq. (2) and Eq. (3) are the state equations for the system.

$$\begin{bmatrix} v_1(n+1)\\ v_2(n+1)\\ v_3(n+1)\\ :\\ :\\ v_{M-1}(n+1)\\ v_M(n+1)\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & .. & 0\\ 0 & 0 & 1 & 0 & 0 & .. & 0\\ 0 & 0 & 0 & 1 & 0 & .. & 0\\ . & . & . & . & . & ... & .\\ . & . & . & . & . & ... & .\\ 0 & 0 & 0 & 0 & 0 & .. & 1\\ a(M) & a(M-1) & a(M-2) & a(M-3) & a(M-4) & .. & a(1)\end{bmatrix} \begin{bmatrix} v_1(n)\\ v_2(n)\\ v_3(n)\\ :\\ :\\ v_{M-1}(n)\\ v_M(n)\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}^T\left((x(n)+w(n)\right) \quad (4)$$

$$\Rightarrow v(n+1)=Av(n)+c(x(n)+w(n)) \quad (5)$$

Where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & .. & 0\\ 0 & 0 & 1 & 0 & 0 & .. & 0\\ 0 & 0 & 0 & 1 & 0 & .. & 0\\ . & . & . & . & . & ... & .\\ . & . & . & . & . & ... & .\\ 0 & 0 & 0 & 0 & 0 & .. & 1\\ a(M) & a(M-1) & a(M-2) & a(M-3) & a(M-4) & .. & a(1)\end{bmatrix} \quad (6)$$

and $c=[0\ 0\ 0.\ 0\ 1]^T$ (7)

The output can be computed from the state variables at time n using

$$y(n)=v_{M+1}(n)+\xi(n)=x(n)+w(n)+\begin{bmatrix}a(M) & a(M-1) & \cdots\cdots & a(1)\end{bmatrix}\begin{bmatrix}v_1(n)\\ v_2(n)\\ \vdots\\ v_M(n)\end{bmatrix}+\xi(n)=b^Tv(n)+d(x(n)+w(n))+\xi(n) \quad (8)$$

Where $b=[a(M)\ a(M-1)\ a(1)]^T)$ (9)

and d=1 (10)

The discrete Kalman filter algorithm

State Equation

$$v(n+1)=A(n)v(n)+C(n)(x(n)+w(n))$$

Observation Equation

$$y(n)=B(n)v(n)+d(x(n)+w(n))+\xi(n)$$

The system noise $w(n)$ and the measurement noise $\xi(n)$ are assumed to be white Gaussian noise with known variances $Qw^{(n)}$ and $Q\xi^{(n)}$ respectively.

Initialization: $\hat{v}(0|0)=E\{v(0)\}$

$$P(0|0)=E\{v(0)v^H(0)\}$$

Computation: For $n=1,2$ compute

1. Assuming the estimate of state vector $\hat{v}(n/n)$, the error covariance $P(n/n)$ are obtained from n^{th} iteration. The predicted state vector and prediction error covariance matrix at the $(n+1)^{th}$ iteration are then computed from

$$\hat{v}(n+1|n)=A(n)\hat{v}(n|n)$$

$$P(n+1|n)=A(n)P(n|n)A^H(n)+Qw(n+1)$$

2. The Kalman filter gain $K(n+1)$ is then computed from

$$K(n+1)=P(n+1|n)B^H(n+1[B(n+1)P(n+1)B^H(n+1+Q_\xi(n+1)]^{-1.}$$

3. The estimate of the state vector and the corresponding error covariance matrix are updated after obtaining a new measurement data $y(n+1)$ at time n using.

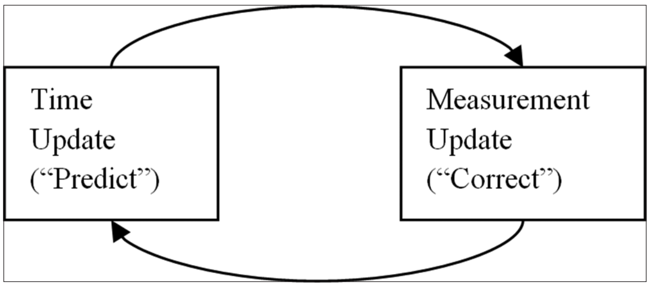

Fig. 1: The ongoing discrete Kalman filter cycle

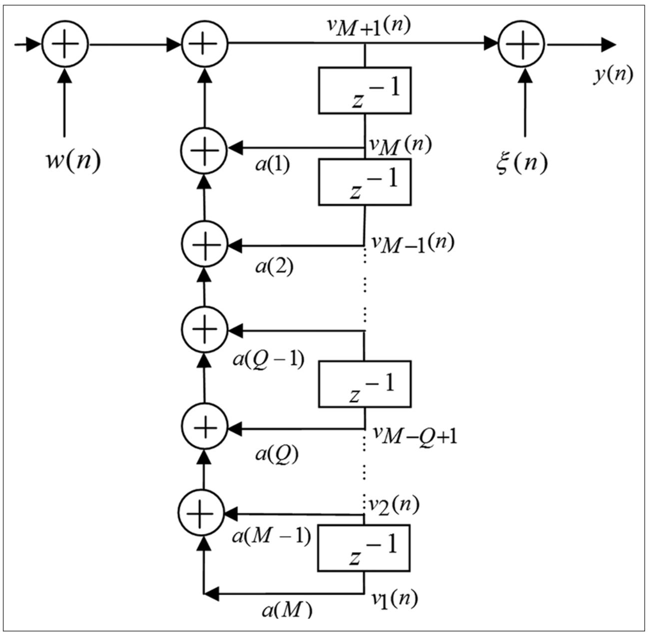

Fig. 2: Direct form II realization of the discrete time system with input-output description

$$\hat{v}(n+1|n+1)=\hat{v}(n+1|n)+K(n+1)\left[Y(n+1)-B(n+1)\hat{v}(n+1|n)\right]$$

$$P(n+1|n+1)=\left[1-K(n+1)B(n+1)\right]P(n+1|n)$$

As the time update projects the current state estimate ahead in time, the measurement update adjusts the projected estimate by an actual measurement at that particular time. The first task during the measurement update is to compute Kalman gain. Next, is to measure the process to obtain the measurement and then generate a posteriori state estimate by incorporating the measurement. Finally, the last step is to obtain the posteriori error covariance estimate. Thus, after each time and measurement update pair, this loop process is repeated to project or predict the new time step priori estimates using the previous time step posteriori estimates. The Kalman filter recursively conditions the current estimate on all of the past measurements. The complete picture of the operation of Kalman filter is shown in Fig. 3.

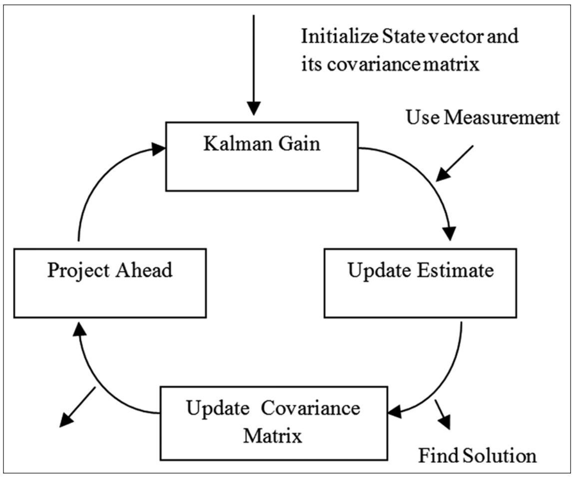

Fig. 3: A complete picture of the operation of the Kalman filter

IMPLEMENTATION AND SIMULATION

Estimating a constant using discrete Kalman filter

1. Example: Let us attempt to estimate a scalar constant x=2, a voltage for example. Let's assume that we have the ability to take measurements of the constant, but that the measurements are corrupted by a √0.0 volt RMS white Gaussian measurement noise.
2. Simulation: The process is governed by the linear difference equation (5) with a measurement given by Eq. (8). The state does not change from step to step so A=1. Though the process noise w=0, a very small process variance of the order of Q_w=0.01 is assumed. Here, the state is nothing but measurement so **C**=1. The variance of measurement noise is considered as Q_ξ=0.1. Let the initial estimate of **v** and error covariance P be 1.50 distinct measurements *y(n)* that had an error normally distributed around zero with a standard deviation of √0.1 is then simulated. Fig. 4 depicts the results of this simulation.

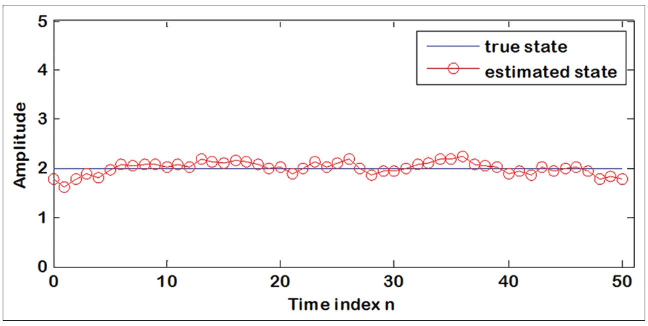

Fig. 4: Estimating a constant using discrete Kalman filter

Estimating a random constant having process noise using discrete Kalman filter

1. Example: Let us attempt to estimate a scalar random constant *x=2* corrupted by √0.1 volt RMS white Gaussian process noise, a voltage for example. The measurements are corrupted by a √0.01 volt RMS white Gaussian measurement noise.
2. Simulation*:* The process is governed by the linear difference equation (5) with a measurement given by Eq. (8). Here, the process noise and measurement noise are considered as white Gaussian noises with variances 0.1 and 0.01 respectively. The state matrix A and the measurement matrix **C** are both taken as 1. Let the initial estimate of **v** and error covariance P be 1.50 distinct measurements *y(n)* that had an error normally distributed around zero with a standard deviation of √0.01 is then simulated. Fig. 5 depicts the results of this simulation.

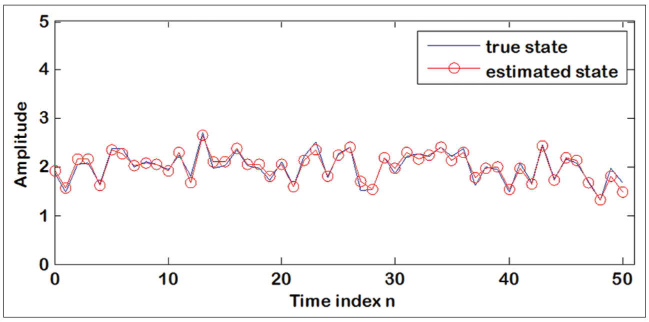

Fig. 5: Estimating a random constant having process noise using discrete Kalman filter

Estimating an AR (p) process using discrete Kalman filter

1. Example*:* Let x(n) be the AR (p) process that is generated by the following difference equation

$$x(n)=\sum_{k=1}^{p} a(k)x(n-k)+w(n) \qquad (11)$$

Where *w(n)* is the white Gaussian noise with a variance 0.36, and let

$$y(n)=x(n)+\xi(n) \qquad (12)$$

be noisy measurements of *x(n)* and ξ(n) is white Gaussian noise with variance 0.01 that is uncorrelated with w(n).

Let p=4 so the AR (4) process is generated according to the difference equation

$$x(n)=0.1x(n-1)+0.2x(n-2)+0.3x(n-3)+0.4x(n-4)+w(n) \qquad (13)$$

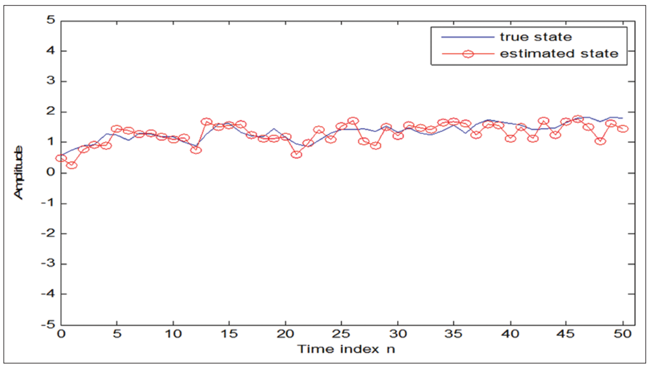

Fig. 6: Estimating an auto regressive (4) process using discrete Kalman filter

The state matrix A is a matrix of order 1×p and the measurement matrix **C** is an identity matrix of order p.

Let the initial estimate of **v** be a zero vector matrix of order 1×p and error covariance P be identity matrix of order p.

2. Simulation*:* The state matrix A is a matrix of order 1×4 and the measurement matrix **C** is an identity matrix of order 4. Let the initial estimate of **v** be a zero vector matrix of order 1×4 and error covariance **P** be identity matrix of order 4. 50 distinct measurements *y(n)* that had an error normally distributed around zero with a standard deviation of $\sqrt{0.01}$ is simulated. Fig. 6 depicts the results of this simulation.

In all the cases, the true value is given by the solid line and the filter estimate by the remaining curve. Under conditions where the covariance of process noise Q_w and measurement noise Q_ξ are constant, both the estimation error covariance P and Kalman gain K will stabilize quickly and then remain constant. If this is the case, these parameters can be pre-computed by running the filter off-line. It is frequently the case however that the measurement error (in particular) does not remain constant.

CONCLUSION

It is tried to estimate the true state by implementing discrete Kalman filter for different cases and observed that the results are satisfactory.

REFERENCES

1. Simon D. Optimal State Estimation Kalman, H Infinity, and Non-linear Approaches. Hoboken: John Wiley and Sons, INC; 2006.
2. Monson H. Hayes, Statistical Digital Signal Processing and Modeling. New York, NY: John Wiley and Sons, INC; 1996.
3. Diniz PS. Adaptive Filtering Algorithms and Practical Implementation. New York: Kluwer Academic Publishers; 2013.
4. Haykin S. Kalman Filtering And Neural Networks. New York: John Wiley & Sons, INC; 2001.
5. Deller JR, Hansen JH, Proakis JG. Discrete Time Processing of Speech Signals. New York: The Institute of Electrical and Electronic Engineers INC; 1993.

Author Queries???

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