ON INTUITIONISTIC FUZZY BI-IDEALS IN GAMMA NEAR RINGS

EZHILMARAN D1*, DHANDAPANI A2

1Department of Mathematics, School of Advanced Sciences, VIT University, Vellore - 632 014, Tamil Nadu, India. 2Department of Mathematics, Salalah College of Technology, Salalah, Sultanate of Oman. Email: ezhil.devarasan@yahoo.com, swetha@gmail.com

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ABSTRACT

The fuzzy set theory developed by Zadeh and others has found many applications in the domain of mathematics. Gamma near-rings were defined by Satyanarayana and the ideal theory in gamma near rings was studied by Satyanarayana and Booth. In this paper, we introduce the intuitionistic fuzzy bi-ideals in \(\Gamma\)-near-rings and investigate some of their related properties.

Keywords: \(\Gamma\)-near-rings, Intuitionistic fuzzy ideals, Intuitionistic fuzzy bi-ideals.

1. INTRODUCTION

Following the introduction of fuzzy sets by Zadeh [20], the fuzzy set theory has been used for many applications in the domain of mathematics and elsewhere. The idea of "Intuitionistic Fuzzy Set (IFS)" was first published by Atanassov [1] as a generalization of the notion of fuzzy set. The concept of \(\Gamma\)-near-ring, a generalization of both the concepts nearring and \(\Gamma\)-ring, was introduced by Satyanarayana [16,17]. Later, several authors such as Booth and Satyanarayana [2,3,5,7-10,14,19] studied the ideal theory of \(\Gamma\)-near-rings. Later Jun et al. [8,9] considered the fuzzification of left (resp. right) ideals of \(\Gamma\)-near-rings. In this paper, we introduce the notion an intuitionistic fuzzy bi-ideal in a \(\Gamma\)-near-ring and some properties of such bi-ideals are investigated. The homomorphic property of intuitionistic fuzzy bi-ideals is established.

2. PRELIMINARIES

In this section, we include some elementary aspects that are necessary for this paper.

Definition 2.1 [16]: A nonempty set \(R\) with two binary operations \("+"\) (addition) and \(".*\) (multiplication) is called a near-ring if it satisfies the following axioms:

i. \((R, +)\) is a group,
ii. \((R, \cdot)\) is a semigroup,
iii. \((x + y).z = x.z + y.z\), for all \(x, y, z \in R\).

Definition 2.2 [17]: A \(\Gamma\)-near-ring is a triple \((M, +, \cdot, \Gamma)\) where,

i. \((M, +)\) is a group,
ii. \((\cdot)\) is a binary operation on \(M\) such that for each \(\alpha \in \Gamma\), \((M, +, \cdot, \alpha)\) is a near-ring,
iii. \(x \cdot (y \cdot z) = (x \cdot y) \cdot z\), for all \(x, y, z \in M\) and \(\alpha, \beta \in \Gamma\).

Definition 2.3 [17]: A subset \(A\) of a \(\Gamma\)-near-ring \(M\) is called a left (resp. right) ideal of \(M\) if

i. \((A, +)\) is a normal divisor of \((M, +)\),
ii. \(u \cdot (x + v) - u \cdot v \in A\) (resp. \(x + u \cdot v \in A\)) for all \(x, \alpha \in A\) and \(u, v \in M\).

Definition 2.4 [18]: Let \(M\) be \(\Gamma\)-near-ring. A subgroup \(A\) of \(M\) is called a bi-ideal of \(M\) if \((A_{\Gamma M}) \cap (A_{M\Gamma}) = A_{\Gamma M}\), where the operation \("**\) is defined by,

\[A**B = (a(y \cdot b) - a\cdot y)/a, a', y' \in A, y' \in \Gamma, b, b' \in B\].

Definition 2.5 [17]: Let \(M\) be \(\Gamma\)-near-ring. A subgroup \(Q\) of \(M\) is called a quasi-ideal of \(M\) if

\[Q(\Gamma M) \cap (MQ)^* \subset Q\].

Definition 2.6 [9]: Let \(M\) and \(N\) be \(\Gamma\)-near-rings. A mapping \(f: M \rightarrow N\) is said to be a homomorphism if \(f(a \cdot b) = f(a) \cdot f(b)\) for all \(a, b \in M\) and \(a, b \in \Gamma\).

Definition 2.7 [9]: A fuzzy set \(\mu\) in a \(\Gamma\)-near-ring \(M\) is called a fuzzy left (resp. right) ideal if \(\mu(x) \geq \mu(a \cdot b)\) for all \(x, a, b \in M\) and \(a \in \Gamma\).

Notation: For the sake of simplicity, we shall use the symbol \(A = <\mu, v_X >\) for the IFS,

\[A = (x, \mu_X, v_X) \forall x \in X\].

Definition 2.10 [11]: Let \(X\) be a nonempty fixed set. An IFS \(A\) in \(X\) is an object having the form \(A = (x, \mu_X, v_X) \forall x \in X\), where the functions \(\mu_X: X \rightarrow [0, 1]\) and \(v_X: X \rightarrow [0, 1]\) denote the degree of membership and degree of nonmembership of each element \(x \in X\) to the set \(A\), respectively, and \(0 \leq \mu_X(x) + v_X(x) \leq 1\).

In what follows, \(M\) will denote a \(\Gamma\)-near-ring unless otherwise specified.

Definition 3.1: An intuitionistic fuzzy ideal \(A = <\mu_A, v_A>\) of \(M\) is called an intuitionistic fuzzy bi-ideal of \(M\) if,
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Example 3.2: Let R be the set of all integers, then R is a ring.

Take M = R = R. Let a, b ∈ M, α, β ∈ R. Suppose αab is the product of a, α, b ∈ R.

Then M is a Γ-near-ring.

Define an IFS A = {a_n, v_m, v_n} in R as follows.

\[ A_1 = \{a_n, v_m, v_n\} \]

Where, \( t \in [0, 1], s \in [0, 1] \) and \( t + s \leq 1 \).

By routine calculations,

Clearly, A is an intuitionistic fuzzy bi-ideal of a Γ-near-ring M.

Lemma 3.3: If A is a bi-ideal of M then for any \( 0 < t, s < 1 \), there exists an intuitionistic fuzzy bi-ideal C = \{a_n, v_m, v_n\} of M such that \( C_{\alpha \beta} = A \).

Proof: Let C → [0, 1] be a function defined by

\[ C_i(x) = \begin{cases} 1 \text{ if } x \in B, \\ 0 \text{ if } x \notin B. \end{cases} \]

Now suppose that B is a bi-ideal of M. For all \( x, y, z \in B \), we have,

\[ \mu_{A}(x-y) \geq \mu_{B}(x) \land \mu_{B}(y), \]

\[ \nu_{A}(x-y) \leq \nu_{B}(x) \lor \nu_{B}(y). \]

Also, for all \( x, y, z \in B \) and \( \alpha, \beta \in \Gamma \) such that \( x\alpha y = z \in B \), we have,

\[ \mu_{A}((x \alpha y \beta z) \land (x \alpha (y + z) - x \alpha z)) = \mu_{A}((x \alpha y) \land \mu_{A}(y) \lor \nu_{A}(y)). \]

\[ (x \alpha (y + z) - x \alpha z)) = \nu_{A}((x \alpha y) \lor \nu_{A}(y)). \]

Thus \( C_{\alpha \beta} \) is an intuitionistic fuzzy bi-ideal of M.

Lemma 3.4: Let B be a nonempty subset of M. Then B is a bi-ideal of M if and only if the IFS B = \{\Lambda_{\alpha \beta} \land \Lambda_{\alpha \beta}\} is an intuitionistic fuzzy ideal of M.

Proof: Let \( x, y \in B \). From the hypothesis, \( x-y \in B \).

i. If \( x, y \in B \), then \( x-y = 0, \Lambda_{\alpha \beta}(x-y) = 1 \) and \( \Lambda_{\alpha \beta}(y) = 0 \). In this case, \( \Lambda_{\alpha \beta}(x-y) = 1 = \Lambda_{\alpha \beta}(y) \).

ii. If \( x \in B, y \notin B \) then \( \Lambda_{\alpha \beta}(x-y) = 0, \Lambda_{\alpha \beta}(x) = 0 \), and \( \Lambda_{\alpha \beta}(y) = 1 \). Thus, \( \Lambda_{\alpha \beta}(x-y) = 0 \leq \Lambda_{\alpha \beta}(x) \lor \Lambda_{\alpha \beta}(y) \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x-y) = 0, \Lambda_{\alpha \beta}(x) = 0 \), and \( \Lambda_{\alpha \beta}(y) = 1 \). Thus, \( \Lambda_{\alpha \beta}(x-y) = 0 \leq \Lambda_{\alpha \beta}(x) \lor \Lambda_{\alpha \beta}(y) \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x-y) = 0, \Lambda_{\alpha \beta}(x) = 0 \), and \( \Lambda_{\alpha \beta}(y) = 0 \). Thus, \( \Lambda_{\alpha \beta}(x-y) = 0 \).

Thus (i) of Definition 3.1 holds good.

Let \( x, y \in B \). From the hypothesis, \( x-y \in B \).

i. If \( x, y \in B \), then \( x-y = 0, \Lambda_{\alpha \beta}(x-y) = 1 \) and \( \Lambda_{\alpha \beta}(y) = 0 \). In this case, \( \Lambda_{\alpha \beta}(x-y) = 1 \).

ii. If \( x \in B, y \notin B \), then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

Thus (ii) of Definition 3.1 holds good.

Let \( x, y, z \in B \) and \( \alpha, \beta \in \Gamma \). From the hypothesis, \( x \alpha y = z \).

i. If \( x, z \in B \), then \( \Lambda_{\alpha \beta}(x) = 1 \).

ii. If \( x \in B, y \notin B \), then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x-y) = 1 \).

Thus (iii) of Definition 3.1 holds good.

Conversely, suppose that IFS B = \{\Lambda_{\alpha \beta}, \Lambda_{\alpha \beta}\} is an intuitionistic fuzzy ideal of M. Then by Lemma 3.3, \( \Lambda_{\alpha \beta} \) is two-valued, hence B is a bi-ideal of M.

This completes the proof.

Theorem 3.5: If \( \{A_{\alpha \beta}\} \) is a family of intuitionistic fuzzy bi-ideals of M then \( \cap \{A_{\alpha \beta}\} \) is an intuitionistic fuzzy bi-ideal of M, where \( \cap \{A_{\alpha \beta}\} \land \Lambda_{\alpha \beta} \).

\( \Lambda_{\alpha \beta}(x) = \inf \{\Lambda_{\alpha \beta}(x) | x \in \Lambda, x \in M \} \land \Lambda_{\alpha \beta}(y) = \sup \{\Lambda_{\alpha \beta}(y) | y \in \Lambda, x \in M \}.

Proof: Let \( x, y \in M \). Then we have,

\[ \Lambda_{\alpha \beta}(x-y) = \inf \{\Lambda_{\alpha \beta}(x-y) | x \in \Lambda, y \in M \} \]

\[ = \inf \{\Lambda_{\alpha \beta}(x-y) \land \Lambda_{\alpha \beta}(x-y) | x \in \Lambda, y \in M \} \]

\[ = \inf \{\Lambda_{\alpha \beta}(x-y) \land \Lambda_{\alpha \beta}(x-y) | x \in \Lambda, y \in M \} \]

\[ = \Lambda_{\alpha \beta}(x-y) \land \Lambda_{\alpha \beta}(x-y). \]

Let \( x, y \in M \). Then we have,

\[ \Lambda_{\alpha \beta}(x+y) = \inf \{\Lambda_{\alpha \beta}(x+y) | x \in \Lambda, y \in M \} \]

\[ = \Lambda_{\alpha \beta}(x+y) \land \Lambda_{\alpha \beta}(x+y). \]

Thus (ii) of Definition 3.1 holds good.

Let \( x, y \in B \). From the hypothesis, \( x+y \in B \).

i. If \( x, y \in B \), then \( x+y = 0, \Lambda_{\alpha \beta}(x+y) = 1 \) and \( \Lambda_{\alpha \beta}(y) = 0 \). In this case, \( \Lambda_{\alpha \beta}(x+y) = 1 \).

ii. If \( x \in B, y \notin B \), then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

Thus (iii) of Definition 3.1 holds good.

Let \( x, y \in B \). From the hypothesis, \( x+y \in B \).

i. If \( x, y \in B \), then \( x+y = 0, \Lambda_{\alpha \beta}(x+y) = 1 \) and \( \Lambda_{\alpha \beta}(y) = 0 \). In this case, \( \Lambda_{\alpha \beta}(x+y) = 1 \).

ii. If \( x \in B, y \notin B \), then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

Thus (iv) of Definition 3.1 holds good.

Let \( x, y \in B \). From the hypothesis, \( x+y \in B \).

i. If \( x, y \in B \), then \( x+y = 0, \Lambda_{\alpha \beta}(x+y) = 1 \) and \( \Lambda_{\alpha \beta}(y) = 0 \). In this case, \( \Lambda_{\alpha \beta}(x+y) = 1 \).

ii. If \( x \in B, y \notin B \), then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iii. If \( x \notin B, y \in B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

iv. If \( x \notin B, y \notin B \) then \( \Lambda_{\alpha \beta}(x+y) = 1 \).

Thus (iv) of Definition 3.1 holds good.
Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. 

$$V_{\Lambda}(\{a \alpha b \beta z\} \cup \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\} = \sup(\{v(x) \vee v(y)\}) | x, y, z \in M$$

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Hence, $\Lambda = \{A : V_{\Lambda}(A)\}$ is an intuitionistic fuzzy bi-ideal of $M$.

**Theorem 3.6:** If $A$ is an intuitionistic fuzzy bi-ideal of $M$ then $A$ is also an intuitionistic fuzzy bi-ideal of $M$.

**Proof:** Let $A(x) = y \in M$. We have,

$$\Lambda(A) = \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\} \cup \{x \alpha y \beta z\}$$

Let $x, y \in M$. We have,

$$\mu_A(x) = \begin{cases} 1 - \mu(x), & \mu(x) \\ \mu(x), & \mu(x) \end{cases}$$

Therefore, $A$ is also an intuitionistic fuzzy bi-ideal of $M$.

**Theorem 3.7:** An IFS $A$ is an intuitionistic fuzzy bi-ideal of $M$ if and only if the level sets $U(\mu) = \{x : \mu(x) \geq t\}$ and $L(\nu) = \{x : \nu(x) \leq t\}$ are a bi-ideal of $M$ when it is non-empty.

**Proof:** Let $B$ be an intuitionistic fuzzy bi-ideal of $M$. Then $\mu_B(x) = \mu_B(x) \wedge \mu_B(x) \geq t$.

$$\mu_B(x) = \begin{cases} 1 - \mu(x), & \mu(x) \\ \mu(x), & \mu(x) \end{cases}$$

Next, define $\mu_B(x) = \begin{cases} 1 - \mu(x), & \mu(x) \\ \mu(x), & \mu(x) \end{cases}$.

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Next, define $\mu_B(x) = \begin{cases} 1 - \mu(x), & \mu(x) \\ \mu(x), & \mu(x) \end{cases}$.

Consequently, $A$ is an intuitionistic fuzzy bi-ideal of $M$.

**Theorem 3.8:** Let $A$ be an intuitionistic fuzzy bi-ideal of $M$. If $M$ is completely regular, then $\mu_A(x) = \mu_A(x)$ and $\nu_A(x) = \nu_A(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

**Proof:** Straightforward.

Let $f$ be mappings from a set $X$ to $Y$, and $A$ be IFS on $Y$. Then the preimage of $\mu$ under $f$, denoted by $f^{-1}(A)$, is defined by:

$$f^{-1}(A) = \{x : f(x) \in A\}$$

for all $x \in X$.

**Theorem 3.9:** Let the pair of mappings $f : M \rightarrow N$ be a homomorphism of $\Gamma$-near-rings.

If $\mu$ is an intuitionistic fuzzy bi-ideal of $N$, then the preimage $f^{-1}(A)$ of $A$ under $f$ is an intuitionistic fuzzy bi-ideal of $M$.

**Proof:** Let $x, y \in M$. Then we have:

$$f^{-1}(\mu(x) = \mu(f(x)) = \mu(f(x-y))$$
Let \( x, y \in M \). Then we have
\[
\begin{align*}
\mu^{-1}_A(y + x - y) &= \mu_A(f(y + x - y)) \\
&\leq \mu_A(f(x)) \\
&= \mu^{-1}_A(x).
\end{align*}
\]
Therefore, \( f^{-1}(A) \) is an intuitionistic fuzzy bi-ideal of \( M \).

**REFERENCES**