

## APPLICATIONS OF GENERALIZED SUMUDU TRANSFORM

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### ABSTRACT

**Objective:** To solve Cauchy Problem for the Wave Equation in One Dimensional Space using the generalized Sumudu transform.

**Materials and Methods:** We use the new integral transform called the Sumudu transform for the extension to a generalized space and we use the method for extension followed by Zemanian[29].

**Results and Conclusion:** In this paper, I have established the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, I have applied it to solve the cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions.

**Keywords:** Sumudu Transform, Generalized Sumudu Transform

### INTRODUCTION

In the literature we can see that many researcher have extended various integral transforms to the distributional space and the space of generalized functions and studied their operational calculus. In early 90's Watugala[23] have introduced a new integral transform namely the Sumudu transform and applied it to solve the physical phenomenon in science and engineering. The fundamental properties of Sumudu transform can be seen in [22-26]. Moreover, the applications of Sumudu transform without resorting to a new frequency domain are presented in [1-8, 10-13]. Watugala [25] have extended the Sumudu transform to two variables with emphasis on solutions to partial differential equations. In [6], we can have the applications to convolution type integral equations along

with a Laplace-Sumudu duality. Sadikali[18] presented the abelian theorem for classical Sumudu transform.

The exhaustive literature survey reveals that many authors have contributed their work in the extension of Sumudu transform to a generalized function and Bohemians [14, 20]. They have defined the generalized Sumudu transform and studied its fundamental properties. The distribution theory provides powerful analytical technique to solve many problems that arises in the applied field. The aim of this paper is to extend the Sumudu transform to a generalized functions and investigate abelian theorems and representation theorem of Sumudu transformable generalized functions.

## 1.1 The Sumudu Transformation

For the function  $f(t)$  the Sumudu transform is defined by the equation[23]

$$\mathbb{S}[f(t)] = G(u) = \int_0^{\infty} e^{-t} f(ut) dt \quad u \in (-\tau_1, \tau_2) \quad (1.1)$$

provided the integral on the RHS exists. The Sumudu transform of functions  $f(t)$  ( $t \geq 0$ ) are come to exists which are piecewise continuous and of exponential order defined over the set  $A=[f(t)/\exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) ]$

The above equation can be reduced to following form with suitable change in the variable

$$\mathbb{S}[f(t)] = G(u) = \frac{1}{u} \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt \quad (1.2)$$

The inverse Sumudu transform of function  $G(u)$  is denoted by symbol  $\mathbb{S}^{-1}[G(u)] = f(t)$  and is defined with Bromwich contour integral[24]

$$\mathbb{S}^{-1}[G(u)] = f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\Pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} G(u) du \quad (1.3)$$

## 2 Generalized Sumudu Transform

In(?), authors has extended the Sumudu transform to certain spaces of distributions. Here we list some required results **Testing function space**  $\mathcal{D}_{a,b}$

Let  $\mathcal{D}_{a,b}$  denotes the space of all complex valued smooth functions  $\phi(t)$  on  $-\infty < t < \infty$  on which the functions  $\gamma_k(\phi)$  defined by

$$\gamma_k(\phi) \triangleq \gamma_{a,b,k}(\phi) \triangleq \text{Sup.}_{0 < t < \infty} |K_{a,b}(t) D^k(t)| < \infty \quad (2.1)$$

where

$$K_{a,b}(t) = \begin{cases} e^{at} & 0 \leq t < \infty \\ e^{bt} & -\infty < t < 0. \end{cases}$$

The space  $\mathcal{D}_{a,b}$  is linear space under the pointwise addition of function and their multiplication by complex numbers. Each  $\gamma_k$  is clearly a seminorm on  $\mathcal{D}_{a,b}$  and  $\gamma_0$  is a norm. We assign the topology generated by the sequence of seminorm  $(\gamma_k)_{k=0}^{\infty}$  there by making it a countably multinormed space. Note that for each fixed  $u$  the kernel  $\frac{1}{u} e^{-\frac{t}{u}}$  as a function of  $t$  is a member of  $\mathcal{D}_{a,b}$  if and only if  $a < \text{Re}(\frac{1}{u}) < b$ . With the usual argument we can show that  $\mathcal{D}_{a,b}$  is complete and hence a Frechet space.  $\mathcal{D}'_{a,b}$  denotes the dual of  $\mathcal{D}_{a,b}$  i.e.  $f$  is member of  $\mathcal{D}'_{a,b}$  if and only if it is continuous linear function on  $\mathcal{D}_{a,b}$ . Thus  $\mathcal{D}'_{a,b}$  is a space of generalized functions. Note that the properties of testing function space  $\mathcal{D}_{a,b}$  will follows from[29].

Now we define the generalized Sumudu Transform. Given a generalized Sumudu transformable generalized function  $f(t)$ , the strip of definition  $\Omega_f$  for  $\mathbb{S}[f]$  is a set in  $\mathbb{C}$  defined by  $\Omega_f \triangleq \{u : \omega_1 < \text{Re}(\frac{1}{u}) < \omega_2\}$  since  $f$  or each  $u \in \Omega_f$  the kernel  $\frac{1}{u} e^{-\frac{t}{u}}$  as a function of  $t$  is a member of

$\mathfrak{D}'_{\omega_1, \omega_2}$ .

For  $f \in \mathfrak{D}'_{\omega_1, \omega_2}$ , we can define the generalized Sumudu transform of  $f(t)$  as conventional function

$$G_f(u) \triangleq \mathbb{S}[f(t)] \triangleq \langle f(t), \frac{1}{u} e^{-\frac{t}{u}} \rangle \quad (2.2)$$

We call  $\Omega_f$  the region (or strip) of definition for  $\mathbb{S}[f(t)]$  and  $\omega_1$  and  $\omega_2$  are the abscissas of definition. Note that the properties like linearity and continuity of generalized Sumudu transform will follow from [29].

### 3 Cauchy Problem for the Wave Equation in one Dimensional Space

In this section, we will give the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, we shall apply it to solve the Cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions. Let  $x$  and  $t$  be one dimensional real variables with  $-\infty < x < \infty$  and  $0 < t < \infty$ , consider the partial differential equation

$$c^2 D_{x^2} u = D_{t^2} u \quad (3.1)$$

This is a homogeneous wave equation. Here  $x$  and  $t$  customarily interpreted as the space and time variables respectively, and  $c$  is a positive real number which represents the speed of the wave.

We shall find the solution of the equation (3.1) of the form  $u = u_t(x)$  where  $u_t(x)$  is a generalized function on  $-\infty < x < \infty$  depending upon the parameter  $t$ . Also note that  $D_{x^2}$  represents the generalized differentiation whereas  $D_{t^2}$  represents the parametric differentiation. [29] We impose the following initial conditions

As  $t \rightarrow 0_+$ ,  $u_t(x)$  converges to  $f(x)$  and  $D_t(u_t(x))$  converges to  $g(x)$ , in  $\mathfrak{D}'$  where  $f(x), g(x)$  are given members of  $\mathfrak{D}'$ .

We assume that  $f, g$  are Sumudu transformable generalized functions whose strips of definition intersect, we shall derive the solution through the use of the Sumudu transform. When we apply the Sumudu transform, we treat  $t$  as a parameter and  $x$  as an independent variable. Thus  $\mathbb{S}[u_t(x)] = U_t(u)$ . Here  $U_t(u)$  is a conventional function of both  $t$  and  $u$  for  $0 < t < \infty$  and  $u$  restricted to an appropriate strip of the definition which we assume is not empty.

On application of Sumudu transform to equation (3.1) it transformed to the equation

$$c^2 \frac{1}{u^2} U_t(u) = D_{t^2} U_t(u) \quad (3.2)$$

whose solution is given by

$$U_t(u) = A(u) e^{-\frac{ct}{u}} + B(u) e^{\frac{ct}{u}} \quad (3.3)$$

Now set  $\mathbb{S}[f(x)] = F(u)$  for  $u \in \Omega_f$  and  $\mathbb{S}[g(x)] = G(u)$  for  $u \in \Omega_g$ . Also assume that the limiting process  $t \rightarrow 0_+$  can be interchanged with Sumudu transform. Using the initial condition and for  $u \in \Omega_f \cap \Omega_g$

$$F(u) = U_t(u) |_{t \rightarrow 0_+} = A(u) + B(u) \quad (3.4)$$

$$G(u) = D_t U_t(u) |_{t \rightarrow 0_+} = \frac{-c}{u} A(u) + \frac{c}{u} B(u) \quad (3.5)$$

on solving these equations for  $A(u)$  and  $B(u)$  we get

$$U_t(u) = \frac{1}{2} [F(u) - \frac{u}{c} G(u)] e^{-\frac{ct}{u}} + \frac{1}{2} [F(u) + \frac{u}{c} G(u)] e^{\frac{ct}{u}} \quad (3.6)$$

The terms  $F(u)e^{-\frac{ct}{u}}$  and  $F(u)e^{\frac{ct}{u}}$  are Sumudu transforms on the strip  $\Omega_f$ . Let  $\mathbb{R}_+$  and  $\mathbb{R}_-$  denotes the half planes  $Re(\frac{1}{u}) > 0$  and  $Re(\frac{1}{u}) < 0$  respectively. If  $\Omega_f \cap \Omega_g$  and  $\mathbb{R}_+$  intersects we choose  $\Omega_u = \Omega_f \cap \Omega_g \cap \mathbb{R}_+$ , so that the inverse Sumudu transform of  $u.G(u)$  is determined by the equation

$$\mathbb{S}[1_+(x) * g(x)] = u.G(u) \quad (3.7)$$

On the other hand if  $\Omega_f \cap \Omega_g$  and  $\mathbb{R}_+$  do not intersects then choose  $\Omega_u = \Omega_f \cap \Omega_g$  and in this region the inverse Sumudu transform of  $u.G(u)$  is determined by the equation

$$\mathbb{S}[-1_+(-x) * g(x)] = u.G(u) \quad (3.8)$$

Set  $h(x) = 1_+(x) * g(x)$  in the first case and  $h(x) = -1_+(-x) * g(x)$  in the second case.

By using the formula  $\mathbb{S}[f(t - \tau)] = e^{-\frac{\tau}{u}} F(u)$ , we have

$$u_t(x) = \frac{1}{2} [f(x - ct) - \frac{1}{c} h(x - ct) + f(x + ct) + \frac{1}{c} h(x + ct)] \quad (3.9)$$

For each fixed  $t$  this is a specific generalized function on  $-\infty < x < \infty$  so that it truly has a sense as a generalized function on  $-\infty < x < \infty$  depending upon the parameter  $t$ .

Now we shall show that the solution obtained above satisfies the differential equation and initial conditions. Let  $f^{(n)}$  denoted the generalized  $n^{th}$  derivative of  $f(x)$ . Let consider

$$\begin{aligned} \langle D_{x^2} f(x - ct), \phi(x) \rangle &= \langle f(x - ct), D_{x^2} \phi(x) \rangle \\ &= \langle f(x), D_{x+ct}^2 \phi(x + ct) \rangle \\ &= \langle f(x), D_{x^2} \phi(x + ct) \rangle \\ &= \langle f^{(2)}(x), \phi(x + ct) \rangle \\ &= \langle f^{(2)}(x - ct), \phi(x) \rangle \end{aligned}$$

This shows that  $D_{x^2} f(x - ct) = f^{(2)}(x - ct)$  in  $\mathcal{D}'$  for any  $\phi \in \mathcal{D}$

Also for any  $\phi \in \mathcal{D}$  and  $\Delta t \rightarrow 0$ , the quantity  $-c < \frac{1}{-c\Delta t} [f(x - ct - c\Delta t) - f(x - ct)], \phi(x) \rangle$  converges to  $-c < f^{(1)}(x - ct), \phi(x) \rangle$  according to Zemanian[29] so that  $D_t f(x - ct) = -c f^{(1)}(x - ct)$  in  $\mathcal{D}'$ . Applying the same procedure with  $f^{(1)}$  shows that

$D_{t^2} f(x - ct) = c^2 f^{(2)}(x - ct)$  in  $\mathcal{D}'$  consequently  $D_{t^2} f(x - ct) = c^2 D_{x^2} f(x - ct)$  in  $\mathcal{D}'$

Similar result can be obtained when  $f(x-ct)$  is replaced by either  $f(x + ct), h(x - ct), h(x + ct)$  Thus solution (3.9) satisfies the differential equation in the required sense.

Now to check the initial condition, for any  $\phi \in \mathcal{D}$  and in the sense of convergence in  $\mathcal{D}$   $\frac{1}{2} [\phi(x - ct) + \phi(x + ct)] \rightarrow \phi(x)$  and

$\frac{1}{2} [\phi(x - ct) - \phi(x + ct)] \rightarrow 0$  as  $t \rightarrow 0_+$  This follows from the fact that  $\phi$  is smooth and of bounded support so that  $\phi$  and each of its derivative are uniformly continuous on  $-\infty < x < \infty$   $\langle u_t(x), \phi(x) \rangle = \frac{1}{2} \langle f(x), \phi(x - ct) + \phi(x + ct) \rangle + \langle h(x), \phi(x - ct) - \phi(x + ct) \rangle \rightarrow \langle f(x), \phi(x) \rangle$  for  $t \rightarrow 0_+$  This shows that the first initial condition is satisfied.

Now  $D_t f(x - ct) = -cf^{(1)}(x - ct)$ ,  $D_t f(x + ct) = cf^{(1)}(x - ct)$

$D_t h(x - ct) = -cg(x - ct)$ ,  $D_t h(x + ct) = cg(x - ct)$

and proceed as above we can easily prove the second initial condition.

#### 4 Conclusion

In this paper, I have established the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, I have applied it to solve the cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions.

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