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APPLICATIONS OF GENERALIZED SUMUDU TRANSFORM

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ABSTRACT

Objective: To solve Cauchy Problem for the Wave Equation in One Dimensional Space using the generalized Sumudu transform.

Materials and Methods: We use the new integral transform called the Sumudu transform for the extension to a generalized space and we use the method for extension followed by Zemanian[29].

Results and Conclusion: In this paper, I have established the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, I have applied it to solve the cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions.

Keywords: Sumudu Transform,Generalized Sumudu Transform

INTRODUCTION

In the literature we can see that many researcher have extended various integral transforms to the distributional space and the space of generalized functions and studied their opera- tional calculus.In early 90's Watugala[23] have introduced a new integral transform namely the Sumudu transform and applied it to solve the physical phenomenon in science and engineer- ing.The fundamental properties of Sumudu transform can be seen in[22–26]. Moreover,the applications of Sumudu transform without resorting to a new frequency domain are presented in[1–8,10–13]. Watugala [25] have extended the Sumudu transform to two variables with em- phasis on solutions to partial differential equations.In[6], we can have the applications to con- volution type integral equations along with a Laplace-Sumudu duality. Sadikali[18]presented the abelian theorem for classical Sumudu transform.

The exhaustive literature survey revels that many authors have contributed their work in the extension of Sumudu transform to a generalized function and Bohemians[14,20]. They have defined the generalized Sumudu transform and studied its fundamental properties. The distri- bution theory provides powerful analytical technique to solve many problems that arises in the applied field. The aim of this paper is to extend the Sumudu transform to a generalized func- tions and investigate abelian theorems and representation theorem of Sumudu transformable generalized functions.

1.1 The Sumudu Transformation

For the function $f(t)$ the Sumudu transform is defined by the equation[23]

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty e^{-t} f(ut)dt \quad u \in (-\tau_1, \tau_2) \tag{1.1}$$

provided the integral on the RHS exists. The Sumudu transform of functions $f(t)$ $(t \geq 0)$ are come to exists which are piecewise continuous and of exponential order defined over the set A=[$f(t)$/∃ M,τ_1, $\tau_2 > 0$,|f(t)| $<$ M $e^{\frac{|t|}{\tau_j}}$, if t $\in (-1)^j \times [0, \infty)$]

The above equation can be reduced to following form with suitable change in the variable

$$\mathbb{S}[f(t)] = G(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)dt \tag{1.2}$$

The inverse Sumudu transform of function $G(u)$ is denoted by symbol $\mathbb{S}^{-1}[G(u)] = f(t)$ and is defined with Bromwich contour integral[24]

$$\mathbb{S}^{-1}[G(u)] = f(t) = \lim_{T\to\infty} \frac{1}{2\Pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st} G(u)du \tag{1.3}$$

2 Generalized Sumudu Transform

In(**?**), authors has extended the Sumudu transform to certain spaces of distributions. Here we list some required results **Testing function space** $\mathfrak{D}_{a,b}$
Let $\mathfrak{D}_{a,b}$ denotes the space of all complex valued smooth functions $\phi(t)$ on $-\infty < t < \infty$ on which the functions $\gamma_k(\phi)$ defined by

$$\gamma_k(\phi) \triangleq \gamma_{a,b,k}(\phi) \triangleq \underset{0<t<\infty}{Sup.} |K_{a,b}(t)D^k(t)| < \infty \tag{2.1}$$

where

$$K_{a,b}(t) = \begin{cases} e^{at} & 0 \leq t < \infty \\ e^{bt} & -\infty < t < 0. \end{cases}$$

The space $\mathfrak{D}_{a,b}$ is linear space under the pointwise addition of function and their multiplication by complex numbers. Each γ_k is clearly a seminorm on $\mathfrak{D}_{a,b}$ and γ_0 is a norm. We assign the topology generated by the sequence of seminorm $(\gamma_k)_{k=0}^\infty$ there by making it a countably multinormed space. Note that for each fixed u the kernel $\frac{1}{u}e^{\frac{-t}{u}}$ as a function of t is a member of $\mathfrak{D}_{a,b}$ if and only if $a < Re(\frac{1}{u}) < b$. With the usual argument we can show that $\mathfrak{D}_{a,b}$ is complete and hence a Frechet space. $\mathfrak{D}'_{a,b}$ denotes the dual of $\mathfrak{D}_{a,b}$ i.e. f is member of $\mathfrak{D}'_{a,b}$ if and only if it is continuous linear function on $\mathfrak{D}_{a,b}$. Thus $\mathfrak{D}'_{a,b}$ is a space of generalized functions. Note that the properties of testing function space $\mathfrak{D}_{a,b}$ will follows from[29].
Now we define the generalized Sumudu Transform. Given a generalized Sumudu transformable generalized function $f(t)$,the strip of definition Ω_f for $\mathbb{S}[f]$ is a set in $\mathbb{C}$ defined by $\Omega_f \triangleq \{u : \omega_1 < Re(\frac{1}{u}) < \omega_2\}$ since f or each $u \in \Omega_f$ the kernel $\frac{1}{u}e^{\frac{-t}{u}}$ as a function of t is a member of

$\mathfrak{D}'_{\omega_1,\omega_2}$.

For $f \in \mathfrak{D}'_{\omega_1,\omega_2}$,we can define the generalized Sumudu transform of $f(t)$ as conventional function

$$G_f(u) \triangleq \mathbb{S}[f(t)] \triangleq < f(t), \frac{1}{u}e^{\frac{-t}{u}} > \tag{2.2}$$

We call Ω_f the region (or strip) of definition for $\mathbb{S}[f(t)]$ and ω_1 and ω_2 are the abscissas of definition. Note that the properties like linearity and continuity of generalized Sumudu transform will follows from[29].

3 Cauchy Problem for the Wave Equation in one Dimensional Space

In this section, we will give the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, we shall apply it to solve the cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions. Let x and t be one dimensional real variables with $-\infty < x < \infty$ and $0 < t < \infty$, consider the partial differential equation

$$c^2 D_{x^2} u = D_{t^2} u \tag{3.1}$$

This is a homogeneous wave equation.Here x and t customarily interpreted as the space and time variables respectively, and c is a positive real number which represents the speed of the wave.

We shall find the solution of the equation (3.1) of the form $u = u_t(x)$ where $u_t(x)$ is a generalized function on $-\infty < x < \infty$ depending upon the parameter t. Also note that D_{x^2} represents the generalized differentiation whereas D_{t^2} represents the parametric differentiation.[29] We impose the following initial conditions

As $t \to 0_+, u_t(x)$ converges to $f(x)$ and $D_t(u_t(x))$ converges to $g(x)$, in $\mathfrak{D}'$ where $f(x), g(x)$ are given members of $\mathfrak{D}'$.

We assume that f, g are Sumudu transformable generalized functions whose strips of definition intersects, we shall derive the solution through the use of the Sumudu transform. When we apply the Sumudu transform, we treat t as a parameter and x as a independent variable. Thus $\mathbb{S}[u_t(x)] = U_t(u)$. Here $U_t(u)$ is a conventional function of both t and u for $0 < t < \infty$ and u restricted to an appropriate strip of the definition which we assume is not empty.

On application of Sumudu transform to equation (3.1) it transformed to the equation

$$c^2 \frac{1}{u^2} U_t(u) = D_{t^2} U_t(u) \tag{3.2}$$

whose solution is given by

$$U_t(u) = A(u)e^{\frac{-ct}{u}} + B(u)e^{\frac{ct}{u}} \tag{3.3}$$

Now set $\mathbb{S}[f(x)] = F(u)$ for $u \in \Omega_f$ and $\mathbb{S}[g(x)] = G(u)$ for $u \in \Omega_g$ Also assume that the limiting process $t \to 0_+$ can be interchanged with Sumudu transform. Using the initial condition and for $u \in \Omega_f \cap \Omega_g$

$$F(u) = U_t(u) \mid_{t \to 0_+} = A(u) + B(u) \tag{3.4}$$

$$G(u) = D_t U_t(u) \mid_{t \to 0_+} = \frac{-c}{u} A(u) + \frac{c}{u} B(u) \tag{3.5}$$

on solving these equations for $A(u)$ and $B(u)$ we get

$$U_t(u) = \frac{1}{2}[F(u) - \frac{u}{c} G(u)] e^{\frac{-ct}{u}} + \frac{1}{2}[F(u) + \frac{u}{c} G(u)] e^{\frac{ct}{u}} \tag{3.6}$$

The terms $F(u)e^{\frac{-ct}{u}}$ and $F(u)e^{\frac{-ct}{u}}$ are Sumudu transforms on the strip Ω_f. Let $\mathbb{R}_+$ and $\mathbb{R}_-$ denotes the half planes $Re(\frac{1}{u}) > 0$ and $Re(\frac{1}{u}) < 0$ respectively. If $\Omega_f \cap \Omega_g$ and $\mathbb{R}_+$ intersects we choose $\Omega_u = \Omega_f \cap \Omega_g \cap \mathbb{R}_+$, so that the inverse Sumudu transform of $u.G(u)$ is determined by the equation

$$\mathbb{S}[1_+(x) * g(x)] = u.G(u) \tag{3.7}$$

On the other hand if $\Omega_f \cap \Omega_g$ and $\mathbb{R}_+$ do not intersects then choose $\Omega_u = \Omega_f \cap \Omega_g$ and in this region the inverse Sumudu transform of $u.G(u)$ is determined by the equation

$$\mathbb{S}[-1_+(-x) * g(x)] = u.G(u) \tag{3.8}$$

Set $h(x) = 1_+(x) * g(x)$ in the first case and $h(x) = -1_+(-x) * g(x)$ in the second case.
By using the formula $\mathbb{S}[f(t-\tau)] = e^{\frac{-\tau}{u}} F(u)$, we have

$$u_t(x) = \frac{1}{2}[f(x-ct) - \frac{1}{c} h(x-ct) + f(x+ct) + \frac{1}{c} h(x+ct)] \tag{3.9}$$

For each fixed t this is a specific generalized function on $\infty < x < \infty$ so that it truly has a sense as a generalized function on $\infty < x < \infty$ depending upon the parameter t.
Now we shall show that the solution obtained above satisfies the differential equation and initial conditions. Let $f^{(n)}$ denoted the generalized n^{th} derivative of $f(x)$. Let consider

$$\begin{aligned} < D_{x^2} f(x-ct), \phi(x) > &=< f(x-ct), D_{x^2}\phi(x) > \\ &=< f(x), D^2_{x+ct}\phi(x+ct) > \\ &=< f(x), D_{x^2}\phi(x+ct) > \\ &=< f^{(2)}(x), \phi(x+ct) > \\ &=< f^{(2)}(x-ct), \phi(x) > \end{aligned}$$

This shows that $D_{x^2} f(x-ct) = f^{(2)}(x-ct)$ in $\mathfrak{D}'$ for any $\phi \in \mathfrak{D}$
Also for any $\phi \in \mathfrak{D}$ and $\Delta t \to 0$, the quantity $-c < \frac{1}{-c\Delta t}[f(x-ct-c\Delta t) - f(x-ct)], \phi(x) >$ converges to $-c < f^{(1)}(x-ct), \phi(x) >$ according to Zemanian[29] so that $D_t f(x-ct) = -cf^{(1)}(x-ct)$ in $\mathfrak{D}'$. Applying the same procedure with $f^{(1)}$ shows that
$D_{t^2} f(x-ct) = c^2 f^{(2)}(x-ct)$ in $\mathfrak{D}'$ consequently $D_{t^2} f(x-ct) = c^2 D_{x^2} f(x-ct)$ in $\mathfrak{D}'$
Similar result can be obtained when f(x-ct) is replaced by either $f(x+ct), h(x-ct), h(x+ct)$
Thus solution (3.9)satisfies the differential equation in the required sense.
Now to check the initial condition, for any $\phi \in \mathfrak{D}$ and in the sense of convergence in $\mathfrak{D}$ $\frac{1}{2}[\phi(x-ct) + \phi(x+ct)] \to \phi(x)$ and
$\frac{1}{2}[\phi(x-ct) - \phi(x+ct)] \to 0$ as $t \to 0_+$ This follows from the fact that ϕ is smooth and of bounded support so that ϕ and each of its derivative are uniformly continuous on $-\infty < x < \infty$
$< u_t(x), \phi(x) >= \frac{1}{2} < f(x), \phi(x-ct) + \phi(x+ct) > + < h(x), \phi(x-ct) - \phi(x+ct) > \to < f(x), \phi(x) >$ for $t \to 0_+$ This shows that the first initial condition is satisfied.

Now $D_t f(x-ct) = -cf^{(1)}(x-ct), D_t f(x+ct) = cf^{(1)}(x-ct)$
$D_t h(x-ct) = -cg(x-ct), D_t h(x+ct) = cg(x-ct)$
and proceed as above we can easily prove the second initial condition.

4 Conclusion

In this paper, I have established the application of generalized Sumudu transform. To illustrate the applications of generalized Sumudu transform, I have applied it to solve the cauchy problem for the wave equation in one dimensional space where the initial conditions are generalized functions.

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